

Some results on L -complete lattices

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Abstract

The paper deals with special types of L -ordered set, L -fuzzy complete lattices, and fuzzy directed complete posets (fuzzy *dcpos*). First, a theorem for constructing monotone maps is proved, a characterization for monotone maps on an L -fuzzy complete lattice is obtained, and it is proved that if f is a monotone map on an L -fuzzy complete lattice $(P; e)$, then $\sqcap S_f$ is the least fixpoint of f . A relation between L -fuzzy complete lattices and fixpoints is found and fuzzy versions of monotonicity, rolling, fusion and exchange rules on L -complete lattices are stated. Finally, we investigate $\text{Hom}(P, P)$, where $(P; e)$ is a fuzzy *dcpo*, and we show that $\text{Hom}(P, P)$ is a fuzzy *dcpo*, the map $\gamma \mapsto \bigwedge_{x \in P} e(x, \gamma(x))$ is a fuzzy directed subset of $\text{Hom}(P, P)$, and we investigate its join.

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1 Introduction

Fixed point theory serves as an essential tool for various branches of mathematical analysis and its applications. There are three main approaches to this theory. The first one is the metric approach in which one makes use of the metric properties of the underlying spaces and self-maps. (A primary example of this approach is Banach's Contraction Mapping Theorem.) The second approach is the topological one in which one utilizes the topological properties of the underlying spaces and continuity of self-maps. (A primary example of this approach is Brouwer's Fixed Point Theorem.) Finally, the third approach is the order-theoretic approach.

Recently, based on complete Heyting algebras and fuzzy L -order relation, Zhang and Xie [21] have defined and studied L -fuzzy complete lattices, which are generalizations of traditional complete lattices. They discussed their properties, showed that they coincide with complete and co-complete categories enriched over the frame L [12], and they proved the Tarski Fixed-Point Theorem for an L -fuzzy complete lattice.

Using complete Heyting algebras, Fan and Zhang [7, 18] studied quantitative domains through fuzzy set theory. Their approach first defines a fuzzy partial order, specifically a degree function, on a non-empty set. Then they define and study fuzzy directed subsets and (continuous) fuzzy directed complete posets (*dcpos* for short). Moreover, Yao [16] and Yao and Shi [17] pursued an investigation on quantitative domains via fuzzy sets. They defined the notions of fuzzy Scott topology on fuzzy *dcpos*, Scott convergence and topological convergence for stratified L -filters and study them. They showed that the category of fuzzy *dcpos* with fuzzy Scott continuous maps is Cartesian-closed.

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In [19], Zhang and Liu defined a kind of an L -frame by a pair (A, i_A) , where A is a classical frame and $i_A : L \rightarrow A$ is a frame morphism. For a stratified L -topological space (X, δ) , the pair (δ, i_X) is one of this kind of L -frames, where $i_X : L \rightarrow \delta$, is a map which sends $a \in L$ to the constant map with the value a . Conversely, a point of an L -frame (A, i_A) is a frame morphism $p : (A, i_A) \rightarrow (L, id_L)$ satisfying $p \circ i_A = id_L$ and $Lpt(A)$ denotes the set of all points of (A, i_A) . Then $\{\Phi_x : Lpt(A) \rightarrow L \mid \forall p \in Lpt(A), \Phi_x(p) = p(x)\}$ is a stratified L -topology on $Lpt(A)$. By these two assignments, Zhang and Liu constructed an adjunction between $SL - Top$ and $L - Loc$ and consequently they established the Stone Representation Theorem for distributive lattices by means of this adjunction. They pointed out that, from the viewpoint of lattice theory, Rodabaugh's fuzzy version of the Stone representation theory is just one and it has nothing different from the classical one. While in our opinion, Zhang-Liu's L -frames preserve many features and also seem to have no strong difference from a crisp one.

In [14], Yao introduced an L -frame by an L -ordered set equipped with some further conditions. It is a complete L -ordered set with the meet operation having a right fuzzy adjoint. They established an adjunction between the category of stratified L -topological spaces and the category of L -locales, the opposite category of this kind of L -frames. Moreover, Yao and Shi, [13], defined on fuzzy $dcpo$ s an L -topology, called the fuzzy Scott topology, and then they studied its properties. They defined Scott convergence, topological convergence for stratified L -filters and showed that a fuzzy $dcpo$ is continuous if and only if, for any stratified L -filter, the fuzzy Scott convergence coincides with convergence with respect to the fuzzy Scott topology.

In the mid-1950's Tarski [11] published an interesting result: Every complete lattice has the fixed point property, that is, every order preserving mapping has a fixpoint. Davis [6] proved the converse: Every lattice with the fixed point property is complete. Tarski's Fixpoint Theorem generalizes to CPO (an abbreviation for a \vee -complete poset with a bottom element), i.e. if $f : P \rightarrow P$ is an order preserving map and P is a CPO , then the set of fixpoints of f , $Fix(f)$, is a CPO and so $Fix(f)$ has a least element. But, what is the relationship between the least element of f and other points of P ? The main purpose of this paper is to answer to this question.

The present paper is organized as follows. In Section 2, we list some preliminary notions and results that will be used in the paper. In Section 3, we consider L -fuzzy complete lattices. We show that if f is a monotone map on an L -fuzzy complete lattice $(P; e)$, then $\sqcap S_f$ and $\sqcup T_f$ are the least and greatest fixpoint of f and so we find a relation between these elements and other points of P . Also, we show that every L -complete lattice is, up to isomorphism, an L -complete lattice of fixpoints and we propose fuzzy versions of monotonicity, rolling, fusion and exchange rules on L -complete lattices. In Section 4, we define the concept of a t -fixpoint and we prove that if $(P; e)$ is a fuzzy $dcpo$, then H_P , the set of monotone maps on $(P; e)$, is a fuzzy $dcpo$. We use it to find some of t -fixpoints of f . Finally, we find conditions under which $\sqcap S_f$ exists.

2 Preliminaries

We start with some notions from [5, 7]. A non-empty subset D of a poset $(P; \leq)$ is called *directed* if, for each pair of elements $x, y \in D$, there exists $z \in D$ such that $x, y \leq z$. We say that a poset $(P; \leq)$ is a *pre-COP* or a *dcpo* (an abbreviation for a directed complete partially ordered set) if, for each directed subset D of P , the join of D , $\bigvee D$ (the least upper bound of D in L) exists. A *dcpo* $(P; \leq)$ is called a *CPO* (an abbreviation for a complete partially ordered set) if P has a bottom element.

Let $(L; \vee, \wedge, 0, 1)$ be a bounded lattice. For $a, b \in L$, we say that $c \in L$ is a *relative pseudocomplement* of a with respect to b if c is the largest element with $a \wedge c \leq b$ and we denote it by $a \rightarrow b$. A lattice $(L; \vee, \wedge)$ is said to be a *Heyting algebra* if the relative pseudocomplement $a \rightarrow b$ exists for all elements $a, b \in L$. A *frame* is a complete lattice $(L; \vee, \wedge)$ satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee_{s \in S} (a \wedge s)$ for every $a \in L$ and $S \subseteq L$. It is well known that L is a frame if and only if it is a complete Heyting algebra. In fact, if $(L; \vee, \wedge)$ is a frame, then for each $a, b \in L$, the relative pseudocomplement of a with respect to b , is the element $a \rightarrow b := \bigvee \{x \in L \mid a \wedge x \leq b\}$. In the following, we list some important properties of

complete Heyting algebras, for more details relevant to frames and Heyting algebras, we refer to [9] and [3, Section 7]:

- (i) $(x \wedge y) \rightarrow z = x \rightarrow (y \rightarrow z)$;
- (ii) $x \rightarrow (\bigwedge Y) = \bigwedge_{y \in Y} (x \rightarrow y)$;
- (iii) $(\bigvee Y) \rightarrow z = \bigwedge_{y \in Y} (y \rightarrow z)$.

From now on, in this paper, $(L; \vee, \wedge, 0, 1)$ or simply L always denotes a frame and L^X denotes the set of all maps from a set X into L .

Definition 2.1. [1, 2, 18, 20] Let P be a set and $e : P \times P \rightarrow L$ be a map. The pair $(P; e)$ is called an *L-ordered set* if, for all $x, y, z \in P$, we have

- (E1) $e(x, x) = 1$;
- (E2) $e(x, y) \wedge e(y, z) \leq e(x, z)$;
- (E3) $e(x, y) = e(y, x) = 1$ implies $x = y$.

Proposition 2.2. [16, Prop. 3.7] Let $(X; e)$ be an *L-ordered set*. Then for each $x, y \in X$,

$$e(x, y) = \bigwedge_{z \in X} (e(z, x) \rightarrow e(z, y)) = \bigwedge_{z \in X} (e(y, z) \rightarrow e(x, z)).$$

In an *L-ordered set* $(P; e)$, the map e is called an *L-order relation* on P . If $(P; \leq)$ is a classical poset, then $(P; \chi_{\leq})$ is an *L-ordered set*, where χ_{\leq} is the characteristic function of \leq . We usually denote this *L-ordered set* by $(P; e_{\leq})$. Moreover, for each *L-ordered set* $(P; e)$, the set $\leq_e = \{(x, y) \in P \times P \mid e(x, y) = 1\}$ is a crisp partial order on P and $(P; \leq_e)$ (if there is no ambiguity we write $(P; \leq)$) is a poset. Assume that $(P; e)$ is an *L-ordered set* and $\phi \in L^P$. Define $\downarrow \phi \in L^P$ and $\uparrow \phi \in L^P$, see [8, 18], as follows:

$$\downarrow \phi(x) = \bigvee_{x' \in P} (\phi(x') \wedge e(x, x')), \quad \uparrow \phi(x) = \bigvee_{x' \in P} (\phi(x') \wedge e(x', x)), \quad \forall x \in P.$$

Definition 2.3. [14, 16] A map $f : (P; e_P) \rightarrow (Q; e_Q)$ between two *L-ordered sets* is called *monotone* if for all $x, y \in P$, $e_P(x, y) \leq e_Q(f(x), f(y))$.

A monotone map $f : (P; e_P) \rightarrow (Q; e_Q)$ is called an *L-order isomorphism* if f is one-to-one, onto and $e_P(x, y) = e_Q(f(x), f(y))$ for all $x, y \in P$.

Definition 2.4. [15, 16] Let $(P; e_P)$ and $(Q; e_Q)$ be two *L-ordered sets* and $f : P \rightarrow Q$ and $g : Q \rightarrow P$ be two monotone maps. The pair (f, g) is called a *fuzzy Galois connection* between P and Q if $e_Q(f(x), y) = e_P(x, g(y))$ for all $x \in P$ and $y \in Q$, where f is called the *fuzzy left adjoint* of g , and dually, g is called the *fuzzy right adjoint* of f .

Theorem 2.5. [15, Thm. 3.2] A pair (f, g) is a *fuzzy Galois connection* on (X, e_X) and (Y, e_Y) if and only if both f and g are monotone and (f, g) is a (crisp) *Galois connection* on (X, \leq_{e_X}) and (Y, \leq_{e_Y}) .

Definition 2.6. [16, 15] Let $(P; e)$ be an *L-ordered set* and $S \in L^P$. An element x_0 is called a *join* (respectively, a *meet*) of S , in symbols $x_0 = \sqcup S$ (respectively, $x_0 = \sqcap S$) if, for all $x \in P$,

- (J1) $S(x) \leq e(x, x_0)$ (respectively, (M1), $S(x) \leq e(x_0, x)$);
- (J2) $\bigwedge_{y \in P} (S(y) \rightarrow e(y, x)) \leq e(x_0, x)$ (respectively, (M2), $\bigwedge_{y \in P} (S(y) \rightarrow e(x, y)) \leq e(x, x_0)$).

If a join and a meet of S exist, then they are unique (see [18]).

Theorem 2.7. [18, Thm. 2.2] *Let $(P; e)$ be an L -ordered set, $x_0 \in P$, and $S \in L^P$. Then*

- (i) $x_0 = \sqcup S$ *if and only if* $e(x_0, x) = \bigwedge_{y \in P} (S(y) \rightarrow e(y, x))$ *for all* $x \in P$;
- (ii) $x_0 = \sqcap S$ *if and only if* $e(x, x_0) = \bigwedge_{y \in P} (S(y) \rightarrow e(x, y))$ *for all* $x \in P$.

Definition 2.8. Let $(P; e)$ be an L -ordered set. For all $a \in P$, $\downarrow a : P \rightarrow L$ and $\uparrow a : P \rightarrow L$ are defined by $\downarrow a(x) = e(x, a)$ and $\uparrow a(x) = e(a, x)$, respectively. It can be easily shown that $\sqcup \downarrow a = a$ and $\sqcap \uparrow a = a$ for all $a \in P$ (see [16, Prop 3.16]).

In [21], Zhang et al. an L -fuzzy complete lattice was introduced: An L -ordered set $(P; e)$ is called an *L -fuzzy complete lattice* (or an *L -complete lattice*, for short) if, for all $S \in L^P$, $\sqcap S$ and $\sqcup S$ exist. If $(P; e)$ is an L -complete lattice, then $(P; \leq_e)$ is a complete lattice, where $\vee S = \sqcup \chi_S$ and $\wedge S = \sqcap \chi_S$ for any $S \subseteq P$.

Theorem 2.9. [21, Thm. 2.20] *Let X be a non-empty set. Then $(L^X; \tilde{e})$ is an L -complete lattice, where $\tilde{e}(f, g) = \bigwedge_{x \in X} (f(x) \rightarrow g(x))$ for all $f, g \in L^X$.*

Suppose that X and Y are two sets. For each mapping $f : X \rightarrow Y$, we have a mapping $f^\rightarrow : L^X \rightarrow L^Y$, defined by

$$(\forall y \in Y)(\forall A \in L^X) \left(f^\rightarrow(A)(y) = \bigvee \{A(x) \mid x \in X, f(x) = y\} \right).$$

For simplicity, for any $A \in L^X$, we use $f(A)$ instead of $f^\rightarrow(A)$.

Definition 2.10. [16] Let $(P; e)$ be an L -ordered set. An element $x \in P$ is called a *maximal (or minimal) element* of $A \in L^P$, in symbols $x = \max A$ (or $x = \min A$), if $A(x) = 1$ and for all $y \in P$, $A(y) \leq e(y, x)$ (or $A(y) \leq e(x, y)$). It is easy to see that if A has a maximal (or minimal) element, then it is unique.

Definition 2.11. [10, 16] Let $(X; e)$ be an L -ordered set. An element $D \in L^X$ is called a *fuzzy directed subset* of $(P; e)$ if

$$(FD1) \quad \bigvee_{x \in X} D(x) = 1;$$

$$(FD2) \quad \text{for all } x, y \in X, D(x) \wedge D(y) \leq \bigvee_{z \in X} (D(z) \wedge e(x, z) \wedge e(y, z)).$$

An L -ordered set $(X; e)$ is called a *fuzzy dcpo* if every fuzzy directed subset of $(X; e)$ has a join.

3 Fixpoints on L -complete lattices

In this section, monotone maps on L -ordered set play an important role. Thus, in Theorem 3.3, we propose a procedure for constructing monotone maps. Then we show that every L -complete lattice is, up to isomorphism, an L -complete lattice of fixpoints. We find a relation between least and greatest fixpoints of a monotone map f on an L -complete lattice $(P; e)$ with join and meet with some special elements of L^P . Finally, we present a fuzzy version of monotonicity, rolling, fusion and exchange rules on L -complete lattices.

Lemma 3.1. *Let $(L; \vee, \wedge)$ be a frame and $a_i, b_i \in L$ for all $i \in I$. Then*

$$\bigwedge_{i \in I} (a_i \rightarrow b_i) \leq \left(\bigvee_{i \in I} a_i \right) \rightarrow \left(\bigvee_{i \in I} b_i \right).$$

Proof. Let $u := \bigwedge_{i \in I} (a_i \rightarrow b_i)$. Then $u \leq a_i \rightarrow b_i$ for all $i \in I$, so $u \wedge a_i \leq b_i$ for all $i \in I$. It follows that $u \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} (u \wedge a_i) \leq \bigvee_{i \in I} b_i$ and hence $u \leq \left(\bigvee_{i \in I} a_i \right) \rightarrow \left(\bigvee_{i \in I} b_i \right)$. \square

In Theorem 3.3 and Corollary 3.6, we will use the following proposition that has been stated by W. Yao [16, Prop. 3.16]. We did not find a proof for it, also we could not prove it. In Proposition 3.2, we will prove the proposition with an additional condition (indeed, we add “ $\sqcup S$ (resp. $\sqcap S$) exist” in the statement of [16, Prop. 3.16]).

Proposition 3.2. *Let $(P; e)$ be an L -ordered set, $S \in L^P$ and $\sqcup S$ (resp. $\sqcap S$) exist. Then $a = \max S$ (resp. $a' = \min S$) if and only if $S(a) = 1$ and $a = \sqcup S$ (resp. $S(a') = 1$ and $a' = \sqcap S$).*

Proof. Let $a = \max S$ and $b = \sqcup S$ for some $b \in P$. Then $S(a) = 1$ and $S(y) \leq e(y, a)$ for all $y \in P$. By (J1), $1 = S(a) \leq e(a, b)$. Also, by (J2), $e(b, a) = \bigwedge_{y \in P} (S(y) \rightarrow e(y, a)) = \bigwedge_{y \in P} 1 = 1$, so $a = b$. The other part can be proved in a similar way. \square

Theorem 3.3. *Let $(P; e)$ be an L -ordered set, $(Q; e')$ be an L -complete lattice and $f : P \rightarrow Q$ be a map. For each $x \in P$, we define $S_x : Q \rightarrow L$, by $S_x(y) = \bigvee_{\{z \in P \mid f(z)=y\}} e(z, x)$. Let $F : P \rightarrow Q$ be defined by $F(a) = \sqcup S_a$ for all $a \in P$. Then F is monotone. Moreover, f is monotone if and only if $f = F$.*

Proof. First we show that F is monotone, that is $e(a, b) \leq e'(F(a), F(b))$ for all $a, b \in P$. Put $a, b \in P$. Set $u_a = \sqcup S_a$ and $u_b = \sqcup S_b$. By Theorem 2.7, for all $x \in Q$, we have

$$e'(u_a, x) = \bigwedge_{y \in Q} (S_a(y) \rightarrow e'(y, x)), \quad e'(u_b, x) = \bigwedge_{y \in Q} (S_b(y) \rightarrow e'(y, x)).$$

Hence,

$$e'(u_a, u_b) = \bigwedge_{y \in Q} (S_a(y) \rightarrow e'(y, u_b)) \tag{3.1}$$

$$S_b(y) \leq e'(y, u_b) \text{ for all } y \in Q. \tag{3.2}$$

From (3.2), it follows that $S_a(y) \rightarrow S_b(y) \leq S_a(y) \rightarrow e'(y, u_b)$ for all $y \in Q$ and so by (3.1), $\bigwedge_{y \in Q} (S_a(y) \rightarrow S_b(y)) \leq \bigwedge_{y \in Q} (S_a(y) \rightarrow e'(y, u_b)) = e'(u_a, u_b)$. Also,

$$\begin{aligned} \bigwedge_{y \in Q} (S_a(y) \rightarrow S_b(y)) &= \bigwedge_{y \in f(P)} (S_a(y) \rightarrow S_b(y)), \text{ since } S(a)(y) = 0, \text{ for all } y \in Q - f(P) \\ &= \bigwedge_{y \in P} (S_a(f(y)) \rightarrow S_b(f(y))) \\ &= \bigwedge_{y \in P} \left(\left(\bigvee_{f(z)=f(y)} e(z, a) \right) \rightarrow \left(\bigvee_{f(w)=f(y)} e(w, a) \right) \right) \\ &\geq \bigwedge_{y \in P} \left(\bigwedge_{f(z)=f(y)} (e(z, a) \rightarrow e(z, b)) \right), \text{ by Lemma 3.1} \\ &= \bigwedge_{z \in P} (e(z, a) \rightarrow e(z, b)) \\ &= e(a, b), \text{ by Proposition 2.2} \end{aligned}$$

so $F : (P; e) \rightarrow (Q; e')$ is monotone. Now, we show that, if f is monotone, then $f = F$. Suppose that $f : (P; e) \rightarrow (Q; e')$ is monotone. Put $a \in P$. For all $y \in Q$, $S_a(y) = \bigvee_{\{z \in P \mid f(z)=y\}} e(z, a)$, so $S_a(f(a)) = 1$. Since f is monotone, $f(z) \in y$ implies that $e(z, a) \leq e'(f(z), f(a)) = e'(y, f(a))$, hence $S_a(y) = \bigvee_{\{z \in P \mid f(z)=y\}} e(z, a) \leq e'(y, f(a))$. By Proposition 3.2, $\sqcup S_a = f(a)$ and so $F(a) = f(a)$ for all $a \in P$. \square

Lemma 3.4. *Suppose that $(P; e)$ is an L -ordered set and $S : P \rightarrow L$ is a map such that $\sqcup S$ ($\sqcap S$) exists. Then for each $x \in P$, $S(x) = 1$ implies that $x \leq \sqcup S$ ($\sqcap S \leq x$).*

Proof. Let $a = \sqcup S$ and $u \in P$ such that $S(u) = 1$. Then by Theorem 2.7, $e(a, x) = \bigwedge_{y \in P} (S(y) \rightarrow e(y, x))$ for all $x \in P$, hence $S(y) \leq e(y, a)$ for all $y \in P$. It follows that $1 = S(u) \leq e(u, a)$ and so $u \leq a$. The proof of the other part is similar. \square

Suppose that $(P; e)$ is an L -ordered set and $f : P \rightarrow P$ is a map. Define three maps $S_f : P \rightarrow L$, $T_f : P \rightarrow L$ and $M_f : P \rightarrow L$ by $S_f(x) = e(f(x), x)$, $T_f(x) = e'(x, f(x))$ and $M_f(x) = S_f(x) \wedge T_f(x)$ for all $x \in P$. Moreover, by $\text{Fix}(f)$ we denote the set of all fixpoints of f , that is, $\text{Fix}(f) = \{x \in P \mid f(x) = x\}$ and every point $x \in \text{Fix}(f)$ is said to be a *fixpoint* of f .

Consider the assumptions of Theorem 3.3. Let $\text{Hom}(P, Q)$ be the set of all monotone maps from P to Q . Clearly, $(Q^P; \varepsilon')$ and $(\text{Hom}(P, Q); \varepsilon')$ are L -ordered sets, where $\varepsilon'(\alpha, \beta) = \bigwedge_{x \in P} e'(\alpha(x), \beta(x))$ for all $\alpha, \beta \in Q^P$. It can be easily seen that, the map, $\phi : Q^P \rightarrow \text{Hom}(P, Q)$, sending f to F (see the notations in Theorem 3.3) is a monotone map and $\text{Fix}(\phi) = \text{Hom}(P, Q)$.

Theorem 3.5. *Let $(P; e)$ be an L -complete lattice and $f : (P; e) \rightarrow (P; e)$ be a monotone map. Then $\sqcap S_f$ and $\sqcup T_f$ are fixpoints of f . Indeed, $\sqcap S_f$ is the least fixpoint and $\sqcup T_f$ is the greatest fixpoint of f .*

Proof. Let $\sqcup T_f = a$ and $\sqcap S_f = b$. Then $e(a, x) = \bigwedge_{y \in P} (T_f(y) \rightarrow e(y, x))$ and $e(x, b) = \bigwedge_{y \in P} (S_f(y) \rightarrow e(x, y))$ for all $x \in P$ and so $T_f(y) \leq e(y, a)$ for all $x \in P$. Since f is a monotone map, then $e(y, a) \leq e(f(y), f(a))$ and hence $T_f(y) \leq e(y, f(y)) \wedge e(f(y), f(a)) \leq e(y, f(a))$ for all $y \in P$. Thus by Theorem 2.7, $e(a, f(a)) = \bigwedge_{y \in P} (T_f(y) \rightarrow e(y, f(a))) = 1$. Also, $1 = e(a, f(a)) \leq e(f(a), f(f(a))) = T_f(f(a))$, so by Lemma 3.4, $f(a) \leq a$. Therefore, $f(a) = a$ and a is a fixpoint of f . Now, let u be another fixpoint of f , then $1 = e(u, f(u)) = T_f(u)$, hence by Lemma 3.4, $u \leq a$, whence a is the greatest fixpoint of f . By a similar way, we can show that b is the least fixpoint of f . \square

Corollary 3.6. *Let $(P; e)$ be an L -complete lattice and $f : (P; e) \rightarrow (P; e)$ be a monotone map. Then $\max T_f$ and $\min S_f$ exist and $\max T_f = \sqcup T_f = \sqcup M_f$ and $\min S_f = \sqcap S_f = \sqcap M_f$.*

Proof. By Theorem 3.5 and Proposition 3.2, it can be easily obtained that $\max T_f = \sqcup T_f$ and $\min S_f = \sqcap S_f$. Let $a = \min S_f$ and $b = \max T_f$. By Theorem 3.5, $M_f(a) = 1 = M_f(b)$. Also, for all $y \in P$, we have $M_f(y) \leq S_f(y) \leq e(y, a)$ and $M_f(y) \leq T_f(y) \leq e(y, a)$, so by definition, $\min M_f = a$ and $\max M_f = b$. Now, from [16, Prop. 3.16] we conclude that $\sqcup M_f = b$ and $\sqcap M_f = a$. \square

By [21, Thm. 2.29], we know that, if $(X; e)$ is an L -complete lattice and $f : X \rightarrow X$ is a monotone map, then $\text{Fix}(f)$ is an L -complete lattice. In the next theorem, we will show that each L -complete lattice is of this form. That is, any L -complete lattice is L -isomorphic to $\text{Fix}(f)$ for some suitable monotone map f on a suitable L -complete lattice.

Theorem 3.7. *Let $(P; e)$ be an L -complete lattice. Define $f : (L^P, \tilde{e}) \rightarrow (L^P, \tilde{e})$ by $f(S) = \downarrow \sqcup S$ for each $S \in L^P$. Then f is monotone and there exists an L -order isomorphism between $(\text{Fix}(f); \tilde{e})$ and $(P; e)$.*

Proof. By [21, Thm. 2.29], $(L^P; \tilde{e})$ is an L -complete lattice. First, we show that f is monotone (clearly, f is well defined). Let $S, T \in L^P$ and $x \in P$. Then by Theorem 2.7,

$$e(\sqcup S, x) = \bigwedge_{y \in P} (S(y) \rightarrow e(y, x)), \quad e(\sqcup T, x) = \bigwedge_{y \in P} (T(y) \rightarrow e(y, x)) \quad (3.3)$$

and so

$$\begin{aligned} e(\downarrow \sqcup S, \downarrow \sqcup T) &= \bigwedge_{y \in P} ((\downarrow \sqcup S)(y) \rightarrow (\downarrow \sqcup T)(y)) = \bigwedge_{y \in P} (e(y, \sqcup S) \rightarrow e(y, \sqcup T)) \\ &= e(\sqcup S, \sqcup T), \text{ by Proposition 2.2} \\ &\geq \bigwedge_{y \in P} (S(y) \rightarrow e(y, \sqcup T)), \text{ by (3.3).} \end{aligned}$$

Also, by (3.3), for all $y \in P$, $T(y) \leq e(y, \sqcup T)$, so $S(y) \rightarrow T(y) \leq S(y) \rightarrow e(y, \sqcup T)$ for all $y \in P$ which implies that $\tilde{e}(S, T) = \bigwedge_{y \in P} (S(y) \rightarrow T(y)) \leq \bigwedge_{y \in P} (S(y) \rightarrow e(y, \sqcup T))$. By summing up the above results, it follows that $\tilde{e}(S, T) \leq \tilde{e}(\downarrow \sqcup S, \downarrow \sqcup T)$. That is, $f : (L^P, \tilde{e}) \rightarrow (L^P, \tilde{e})$ is monotone. Define $\alpha : P \rightarrow \text{Fix}(f)$ by $\alpha(x) = \downarrow x$ for all $x \in P$. We know that $(\text{Fix}(f); \tilde{e})$ is an L -complete lattice. Clearly, α is one-to-one. Put $S \in \text{Fix}(f)$. Then $\downarrow \sqcup S = f(S) = S$, so $S \in \text{Im}(\alpha)$. That is, α is onto. Moreover, by Proposition 2.2, for all $a, b \in P$, $\tilde{e}(\alpha(a), \alpha(b)) = \tilde{e}(\downarrow a, \downarrow b) = \bigwedge_{y \in P} (e(y, a) \rightarrow e(y, b)) = e(a, b)$. Therefore, α is an L -order isomorphism. \square

We know that if $(P; \leq)$ is a complete lattice and $f, g : P \rightarrow P$ are two ordered preserving maps such that $F(x) \leq G(x)$ for all $x \in P$, then $\mu_f \leq \mu_g$, where μ_f and μ_g are the least fixpoints of f and g , respectively (see [5, Section 8]). In the next theorem, we generalize this result for L -complete lattices.

Theorem 3.8. (Monotonicity rule.) *Let $(P; e)$ be an L -complete lattice and $f, g : (P; e) \rightarrow (P; e)$ be monotone maps such that $\sqcap S_f = a$ and $\sqcap S_g = b$. Then $\bigwedge_{y \in P} e(f(y), g(y)) \leq e(a, b)$.*

Proof. Since $\sqcap S_f = a$ and $\sqcap S_g = b$, then for each $x \in P$, we have

$$e(x, a) = \bigwedge_{y \in P} (e(f(y), y) \rightarrow e(x, y)), \quad (3.4)$$

$$e(x, b) = \bigwedge_{y \in P} (e(g(y), y) \rightarrow e(x, y)). \quad (3.5)$$

By (3.5), $e(a, b) = \bigwedge_{y \in P} (e(g(y), y) \rightarrow e(a, y))$. Also, by (3.4), $e(f(y), y) \leq e(a, y)$ for all $y \in P$, so $e(g(y), y) \rightarrow e(f(y), y) \leq e(g(y), y) \rightarrow e(a, y)$ for all $y \in P$ and hence, $e(a, b) = \bigwedge_{y \in P} (e(g(y), y) \rightarrow e(a, y)) \geq \bigwedge_{y \in P} (e(g(y), y) \rightarrow e(f(y), y))$. Now, we claim that $e(g(y), y) \rightarrow e(f(y), y) \geq e(f(y), g(y))$ for all $y \in P$. In order to show that our claim is true, it suffices to prove that $e(f(y), g(y)) \wedge e(g(y), y) \leq e(f(y), y)$, which clearly hold by (E3). Hence, our claim is true and so $e(a, b) = \bigwedge_{y \in P} (e(g(y), y) \rightarrow e(f(y), y)) \geq \bigwedge_{y \in P} e(f(y), g(y))$. \square

Theorem 3.9. (Rolling rule.) *Let $(P; e)$ and $(Q; e')$ be L -complete lattices and $f : (P; e) \rightarrow (Q; e')$, $g : (Q; e') \rightarrow (P; e)$ be monotone maps. Then the following hold:*

- (i) $g(\sqcap S_{f \circ g}) = \sqcap S_{g \circ f}$.
- (ii) $\sqcap(g(S_{f \circ g})) = g(\sqcap S_{g \circ f})$.

Proof. (i) The proof of this part follows from Theorem 3.5, and the Rolling Rule from [5, 8.29].

(ii) Let $\sqcap(g(S_{f \circ g})) = b$ and $a = \sqcap S_{g \circ f}$. Then for each $x \in P$,

$$e(x, b) = \bigwedge_{y \in P} (g(S_{f \circ g})(y) \rightarrow e(x, y)) = \bigwedge_{y \in P} ((\bigvee_{\{z \in Q \mid g(z)=y\}} S_{f \circ g}(z)) \rightarrow e(x, y)) \quad (3.6)$$

$$= \bigwedge_{y \in P} \bigwedge_{\{z \in Q \mid g(z)=y\}} (S_{f \circ g}(z) \rightarrow e(x, y)) \quad (3.7)$$

$$= \bigwedge_{y \in Q} (S_{f \circ g}(y) \rightarrow e(x, g(y))). \quad (3.8)$$

$$e(x, a) = \bigwedge_{y \in P} (S_{g \circ f}(y) \rightarrow e(x, y)) = \bigwedge_{y \in P} (e(g \circ f(y), y) \rightarrow e(x, y)). \quad (3.9)$$

Since g is monotone, by (3.8) and (3.9), for each $x \in Q$, $e(g(x), b) = \bigwedge_{y \in Q} (S_{f \circ g}(y) \rightarrow e(g(x), g(y))) \geq \bigwedge_{y \in Q} (S_{f \circ g}(y) \rightarrow e(x, y)) = e(x, \sqcap S_{f \circ g})$. Hence, $e(g(\sqcap S_{f \circ g}), b) \geq e(\sqcap S_{f \circ g}, \sqcap S_{f \circ g}) = 1$ and so by (i),

$a = \sqcap S_{g \circ f} = g(\sqcap S_{f \circ g}) \leq b$. Moreover, by (3.8), $e(f \circ g(y), y) = S_{f \circ g}(y) \leq e(b, g(y))$ for all $y \in Q$, so $e(f \circ g(f(a)), f(a)) \leq e(b, g(f(a)))$. By Theorem 3.5, we have $g(f(a)) = a$ (since $a = \sqcap S_{g \circ f}$) and $f \circ g(f(a)) = f(a)$. It follows that $1 = e(f(a), f(a)) = e(f \circ g(f(a)), f(a)) = e(b, a)$. Therefore, $a = b$, and so by (i), the proof of this part is completed. \square

Theorem 3.10. (Fusion rule.) *Let $(P; e)$ and $(Q; e')$ be two L -complete lattices and let $f : P \rightarrow Q$ possess a right adjoint $f' : Q \rightarrow P$. Let $g : P \rightarrow P$ and $h : Q \rightarrow Q$ be monotone. Then*

- (i) $\bigwedge_{y \in P} e'(f \circ g(y), h \circ f(y)) \leq e'(f(\sqcap S_g), \sqcap S_h)$.
- (ii) $e'(h \circ f(\sqcap S_g), f \circ g(\sqcap S_g)) \leq e'(\sqcap S_h, f(\sqcap S_g))$.

Proof. (i) By Theorem 2.5, (f, f') is a fuzzy Galois connection between P and Q , hence

$$\begin{aligned} 1 = e(f'(\sqcap S_h), f'(\sqcap S_h)) &= e'(f(f'(\sqcap S_h)), \sqcap S_h) \\ &\Rightarrow 1 = e'(h(f(f'(\sqcap S_h))), h(\sqcap S_h)), \text{ since } h \text{ is monotone} \\ &\Rightarrow 1 = e'(h(f(f'(\sqcap S_h))), \sqcap S_h), \text{ by Theorem 3.5.} \end{aligned}$$

It follows that

$$\begin{aligned} e'(f \circ g(f'(\sqcap S_h)), h(f(f'(\sqcap S_h)))) &= e'(h(f(f'(\sqcap S_h))), \sqcap S_h) \wedge e'(f \circ g(f'(\sqcap S_h)), h(f(f'(\sqcap S_h)))) \\ &\leq e'(f \circ g(f'(\sqcap S_h)), \sqcap S_h) = e(g(f'(\sqcap S_h)), f'(\sqcap S_h)). \end{aligned}$$

By Theorem 2.7, for each $y \in P$, $e(g(y), y) = S_g(y) \leq e(\sqcap S_g, y)$, so

$$e(g(f'(\sqcap S_h)), f'(\sqcap S_h)) \leq e(\sqcap S_g, f'(\sqcap S_h)) = e'(f(\sqcap S_g), \sqcap S_h).$$

Therefore, $\bigwedge_{y \in P} e'(f \circ g(y), h \circ f(y)) \leq e'(f(\sqcap S_g), \sqcap S_h)$.

(ii) By Theorem 2.7(ii), we know that $S_h(y) \leq e'(\sqcap S_h, y)$ for all $y \in Q$, and so by Theorem 3.5, $e'(\sqcap S_h, f(\sqcap S_g)) \geq S_h(f(\sqcap S_g)) = e'(h(f(\sqcap S_g)), f(\sqcap S_g)) = e'(h(f(\sqcap S_g)), f(g(\sqcap S_g)))$. \square

Note that, in Theorem 3.10, we showed that $e'(f \circ g(f'(\sqcap S_h)), h(f(f'(\sqcap S_h)))) \leq e'(f(\sqcap S_g), \sqcap S_h)$.

Corollary 3.11. (Exchange rule.) *Let $(P; e)$ and $(Q; e')$ be L -complete lattices and $f, g : P \rightarrow Q$ and $h : Q \rightarrow P$ be monotone maps. If f possesses a right adjoint $f' : Q \rightarrow P$, then*

$$e'(f \circ h \circ g(f'(\sqcap S_{g \circ h})), g \circ h \circ f(f'(\sqcap S_{g \circ h}))) \leq e'(\sqcap S_{f \circ h}, \sqcap S_{g \circ h}).$$

Proof. Let $h' : Q \rightarrow Q$ and $g' : P \rightarrow P$ be defined by $h' = g \circ h$ and $g' = h \circ g$. By Theorem 3.10,

$$e'(f \circ g'(f'(\sqcap S_{h'})), h' \circ f(\sqcap S_{h'})) \leq e'(f(\sqcap S_{g'}), \sqcap S_{h'}) = e'(f(\sqcap S_{h \circ g}), \sqcap S_{g \circ h})$$

and by Theorem 3.9(i), $\sqcap S_{h \circ g} = h(\sqcap S_{g \circ h})$, so that $e'(f(\sqcap S_{h \circ g}), \sqcap S_{g \circ h}) = e'(f(h(\sqcap S_{g \circ h})), \sqcap S_{g \circ h}) = S_{f \circ h}(\sqcap S_{g \circ h})$. Also, by Theorem 2.7, for each $y \in Q$, $S_{f \circ h}(y) \leq e'(\sqcap S_{f \circ h}, y)$, so $e'(f \circ g'(f'(\sqcap S_{h'})), h' \circ f(\sqcap S_{h'})) \leq e'(\sqcap S_{f \circ h}, \sqcap S_{g \circ h})$. \square

4 Fuzzy dcpos

In this section, we define the concept of a t -fixpoint and prove that if $(P; e)$ is a fuzzy dcpo, then the set of monotone maps on $(P; e)$ is a fuzzy dcpo. This will serve us in order to find some of the t -fixpoints of f . Finally, we find conditions under which $\sqcap S_f$ exists.

Theorem 4.1. Let $(P; e)$ be a fuzzy dcpo and H_P be the set of all monotone maps on $(P; e)$. Then $(H_P; \bar{e})$ is a fuzzy dcpo, where $\bar{e}(f, g) = \bigwedge_{x \in P} e(f(x), g(x))$ for all $f, g \in H_P$.

Proof. It is easy to see that $(H_P; \bar{e})$ is an L -ordered set. Let $S : H_P \rightarrow L$ be a fuzzy directed subset of $(H_P; \bar{e})$. Then $\bigvee_{f \in H_P} S(f) = 1$ and for each $f, g \in H_P$,

$$S(f) \wedge S(g) \leq \bigvee_{\gamma \in H_P} (S(\gamma) \wedge \bar{e}(f, \gamma) \wedge \bar{e}(g, \gamma)). \quad (4.1)$$

First, we show that $\sqcup S$ exists. That is, there exists a map $\alpha_0 : P \rightarrow P$ such that

$$\bar{e}(\alpha_0, f) = \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow \bar{e}(\gamma, f)) \quad \text{for all } f \in H_P,$$

which is equivalent to

$$\begin{aligned} \bar{e}(\alpha_0, f) &= \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow (\bigwedge_{y \in P} e(\gamma(y), f(y)))) = \bigwedge_{\gamma \in H_P} \bigwedge_{y \in P} (S(\gamma) \rightarrow e(\gamma(y), f(y))) \\ &= \bigwedge_{y \in P} \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow e(\gamma(y), f(y))). \end{aligned}$$

So, it suffices to show that

$$\bigwedge_{y \in P} e(\alpha_0(y), f(y)) = \bigwedge_{y \in P} \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow e(\gamma(y), f(y))) \quad \text{for all } f \in H_P. \quad (4.2)$$

Put $f \in H$. We claim that, for all $y \in P$, there exists an element $u_y \in P$ such that

$$e(u_y, f(y)) = \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow e(\gamma(y), f(y))). \quad (4.3)$$

Let $x \in P$ and $X := \{\gamma(x) | \gamma \in H_P\}$. Define the map $T_x : P \rightarrow L$ by

$$T_x(y) = \begin{cases} 0 & y \in P - X \\ \bigvee \{S(h) | h \in H_P, h(x) = y\} & y \in X. \end{cases}$$

Clearly, T_x is a well-defined map. In the following, we show that T_x is a fuzzy directed subset on $(P; e)$.

(i) $\bigvee_{u \in P} T_x(u) = \bigvee_{u \in X} T_x(u) = \bigvee_{\gamma \in H_P} T_x(\gamma(x)) = \bigvee_{\gamma \in H_P} \bigvee_{\{h \in H_P | h(x) = \gamma(x)\}} S(h) = \bigvee_{h \in H_P} S(h) = 1$ (since $S : H_P \rightarrow L$ is a fuzzy directed set on $(H_P; \bar{e})$).

(ii) Let $u, v \in P$. If $u \in P - X$ or $v \in P - X$, then by definition, $T_x(u) \wedge T_x(v) = 0$ and so $T_x(u) \wedge T_x(v) \leq \bigvee_{z \in P} (T_x(z) \wedge e(u, z) \wedge e(v, z))$. Otherwise, $u = \gamma_1(x)$ and $v = \gamma_2(x)$ for some $\gamma_1, \gamma_2 \in H_P$. It follows that

$$T_x(u) \wedge T_x(v) = \bigvee_{\{h \in H_P | h(x) = \gamma_1(x)\}} (S(h) \wedge S(k)) \quad (4.4)$$

Also, we have

$$\begin{aligned}
\bigvee_{z \in P} (T_x(z) \wedge e(u, z) \wedge e(v, z)) &= \bigvee_{z \in X} (T_x(z) \wedge e(u, z) \wedge e(v, z)) \\
&= \bigvee_{\gamma \in H_P} (T_x(\gamma(x)) \wedge e(u, \gamma(x)) \wedge e(v, \gamma(x))) \\
&= \bigvee_{h \in H_P} \left(\left(\bigvee_{\{h \in H \mid h(x) = \gamma(x)\}} S(h) \right) \wedge e(u, \gamma(x)) \wedge e(v, \gamma(x)) \right) \\
&= \bigvee_{h \in H_P} \bigvee_{\{h \in H \mid h(x) = \gamma(x)\}} (S(h) \wedge e(u, \gamma(x)) \wedge e(v, \gamma(x))) \\
&= \bigvee_{\gamma \in H_P} (S(h) \wedge e(u, \gamma(x)) \wedge e(v, \gamma(x))) \\
&= \bigvee_{\gamma \in H_P} (S(h) \wedge e(\gamma_1(x), \gamma(x)) \wedge e(\gamma_2(x), \gamma(x))) \\
&\geq \bigvee_{\gamma \in H_P} (S(h) \wedge \bar{e}(\gamma_1, \gamma) \wedge \bar{e}(\gamma_2, \gamma))
\end{aligned}$$

so by (4.1), $\bigvee_{z \in P} (T_x(z) \wedge e(u, z) \wedge e(v, z)) \geq S(\gamma_1) \wedge S(\gamma_2)$. Since γ_1 and γ_2 are arbitrary elements of H_P such that $\gamma_1(x) = u$ and $\gamma_2(x) = v$, then $\bigvee_{z \in P} (T_x(z) \wedge e(u, z) \wedge e(v, z)) \geq S(h) \wedge S(k)$ for all $k, h \in H_P$ such that $h(x) = u$ and $k(x) = v$, thus by (4.4), $\bigvee_{z \in P} (T_x(z) \wedge e(u, z) \wedge e(v, z)) \geq \bigvee_{\{h \in H_P \mid h(x) = \gamma_1(x)\}} \bigvee_{\{k \in H_P \mid k(x) = \gamma_2(x)\}} (S(h) \wedge S(k)) = T_x(u) \wedge T_x(v)$.

(i) and (ii) imply that T_x is a fuzzy directed set on $(P; e)$ for all $x \in P$, whence by the assumption, $\sqcup T_x$ exists for all $x \in P$. Let $u_x := \sqcup T_x$ for all $x \in P$. Then for each $x \in P$, we have

$$e(u_x, z) = \bigwedge_{t \in P} (T_x(t) \rightarrow e(t, z)), \quad \text{for all } z \in P. \quad (4.5)$$

Define a map $\alpha_0 : P \rightarrow L$ by $\alpha_0(x) = \sqcup T_x$ for all $x \in P$. By (4.5), for each $f \in H_P$ and $y, z \in P$,

$$e(\alpha_0(y), z) = e(u_y, z) = \bigwedge_{t \in P} (T_y(t) \rightarrow e(t, z)) \quad (4.6)$$

$$= \bigwedge_{\gamma \in H_P} (T_y(\gamma(y)) \rightarrow e(\gamma(y), z)), \text{ by definition of } T_y \quad (4.7)$$

$$= \bigwedge_{\gamma \in H_P} \left(\left(\bigvee_{\{h \in H \mid h(y) = \gamma(y)\}} S(h) \right) \rightarrow e(\gamma(y), z) \right) \quad (4.8)$$

$$= \bigwedge_{\gamma \in H_P} \bigwedge_{\{h \in H \mid h(y) = \gamma(y)\}} (S(h) \rightarrow e(\gamma(y), z)) \quad (4.9)$$

$$= \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow e(\gamma(y), z)). \quad (4.10)$$

It follows that $\bigwedge_{y \in P} e(\alpha_0(y), f(y)) = \bigwedge_{y \in P} \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow e(\gamma(y), f(y)))$ and so (4.2) holds. Hence $\alpha_0 = \sqcup S$. Now, we show that $\alpha_0 : P \rightarrow P$ is a monotone map. Let $x, y \in P$. Also, by (4.10), for all $z \in P$,

$$e(\alpha_0(x), z) = \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow e(\gamma(x), z)), \quad e(\alpha_0(y), z) = \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow e(\gamma(y), z)).$$

so,

$$S(\gamma) \leq e(\gamma(y), \alpha_0(y)) \quad (4.11)$$

$$e(\alpha_0(x), \alpha_0(y)) = \bigwedge_{\gamma \in H_P} (S(\gamma) \rightarrow e(\gamma(x), \alpha_0(y))). \quad (4.12)$$

Hence, by (4.12),

$$\begin{aligned} e(x, y) \leq e(\alpha_0(x), \alpha_0(y)) &\Leftrightarrow e(x, y) \leq S(\gamma) \rightarrow e(\gamma(x), \alpha_0(y)), \text{ for all } \gamma \in H_P \\ &\Leftrightarrow e(x, y) \wedge S(\gamma) \leq e(\gamma(x), \alpha_0(y)), \text{ for all } \gamma \in H_P \\ &\Leftrightarrow S(\gamma) \leq e(x, y) \rightarrow e(\gamma(x), \alpha_0(y)), \text{ for all } \gamma \in H_P. \end{aligned}$$

Since for each $\gamma \in H_P$, $e(x, y) \leq e(\gamma(x), \gamma(y))$, then by Proposition 2.2, $e(\gamma(y), \alpha_0(y)) \leq e(\gamma(x), \gamma(y)) \rightarrow e(\gamma(x), \alpha_0(y)) \leq e(x, y) \rightarrow e(\gamma(x), \alpha_0(y))$. Thus by (4.11), $S(\gamma) \leq e(x, y) \rightarrow e(\gamma(x), \alpha_0(y))$. Therefore, α_0 is monotone and, whence, it belongs to H_P . \square

Theorem 4.2. *Let $(P; e)$ be an fuzzy dcpo, H_P be the set of all monotone maps on $(P; e)$ and $S : H_P \rightarrow L$ be defined by $S(\alpha) = \bar{e}(Id_P, \alpha)$ for all $\alpha \in H_P$. Then $\sqcup S$ exists and belongs to H_P . Moreover, $S(\alpha) \wedge S(\sqcup S) \leq \bar{e}(\alpha \circ \sqcup S, \sqcup S) \wedge \bar{e}(\sqcup S, \alpha \circ \sqcup S)$ for all $\alpha \in H_P$.*

Proof. First, we show that S is a fuzzy directed subset of (H_P, \bar{e}) . Since $Id_P \in H$ and $S(Id_P) = \bar{e}(Id_P, Id_P) = 1$, then $\bigvee_{\gamma \in H_P} S(\gamma) = 1$. Now, we show that $S(f) \wedge S(g) \leq \bigvee_{\alpha \in H_P} (S(\alpha) \wedge \bar{e}(f, \alpha) \wedge \bar{e}(g, \alpha))$ for all $f, g \in H_P$. Put $f, g \in H_P$.

$$S(f) \wedge S(g) = \left(\bigwedge_{x \in P} e(x, f(x)) \right) \wedge \left(\bigwedge_{x \in P} e(x, g(x)) \right) = \bigwedge_{x \in P} (e(x, f(x)) \wedge e(x, g(x))). \quad (4.13)$$

$$\bigvee_{\alpha \in H_P} (S(\alpha) \wedge \bar{e}(f, \alpha) \wedge \bar{e}(g, \alpha)) = \bigvee_{\alpha \in H_P} \left(\left(\bigwedge_{x \in P} e(x, \alpha(x)) \right) \wedge \left(\bigwedge_{x \in P} e(f(x), \alpha(x)) \right) \wedge \left(\bigwedge_{x \in P} e(g(x), \alpha(x)) \right) \right) \quad (4.14)$$

$$= \bigvee_{\alpha \in H_P} \left(\left(\bigwedge_{x \in P} e(x, \alpha(x)) \wedge e(f(x), \alpha(x)) \right) \wedge \left(\bigwedge_{x \in P} e(g(x), \alpha(x)) \right) \right). \quad (4.15)$$

Clearly, $f \circ g \in H_P$ and for each $x \in P$, we have

(i) $e(x, f \circ g(x)) \geq e(x, f(x)) \wedge e(f(x), f \circ g(x)) \geq e(x, f(x)) \wedge e(x, g(x))$ (since f is monotone).

(ii) $e(f(x), f \circ g(x)) \geq e(x, g(x))$, (since f is monotone),

which imply that $\bigwedge_{x \in P} (e(x, f \circ g(x)) \wedge e(f(x), f \circ g(x))) \geq \bigwedge_{x \in P} (e(x, f(x)) \wedge e(x, g(x)))$. Also,

$$\bigwedge_{x \in P} e(g(x), f \circ g(x)) \geq \bigwedge_{x \in P} e(x, f(x)), \quad \text{since } Im(g) \subseteq P$$

so, we have

$$\begin{aligned} \bigwedge_{x \in P} (e(x, f \circ g(x)) \wedge e(f(x), f \circ g(x))) &\wedge \bigwedge_{x \in P} e(g(x), f \circ g(x)) \\ &\geq \bigwedge_{x \in P} (e(x, f(x)) \wedge e(x, g(x))) \wedge \bigwedge_{x \in P} e(x, f(x)) \\ &= \bigwedge_{x \in P} (e(x, f(x)) \wedge e(x, g(x))). \end{aligned}$$

From $f \circ g \in H$, (4.13) and (4.15), it follows that

$$\begin{aligned}
S(f) \wedge S(g) &= \bigwedge_{x \in P} (e(x, f(x)) \wedge e(x, g(x))) \\
&\leq \bigvee_{\alpha \in H_P} \left(\left(\bigwedge_{x \in P} e(x, \alpha(x)) \wedge e(f(x), \alpha(x)) \right) \wedge \left(\bigwedge_{x \in P} e(g(x), \alpha(x)) \right) \right) \\
&= \bigvee_{\alpha \in H_P} (S(\alpha) \wedge \bar{e}(f, \alpha) \wedge \bar{e}(g, \alpha)).
\end{aligned}$$

Therefore, S is a fuzzy directed subset of (H_P, \bar{e}) . By Theorem 4.1, there exists $\beta \in H_P$ such that $\beta = \sqcup S$, hence by Theorem 2.7(i), for each $f \in H_P$, $\bar{e}(\beta, f) = \bigwedge_{\alpha \in H_P} (S(\alpha) \rightarrow \bar{e}(\alpha, f))$, whence $S(\alpha \circ \beta) \leq \bar{e}(\alpha \circ \beta, \beta)$. From (E2), it can be easily obtained that $S(\alpha \circ \beta) = \bar{e}(Id_P, \alpha \circ \beta) \geq \bar{e}(Id_P, \alpha) \wedge \bar{e}(\alpha, \alpha \circ \beta) \geq \bar{e}(Id_P, \alpha) \wedge \bar{e}(Id_P, \beta) = S(\alpha) \wedge S(\beta)$. Thus, $S(\alpha) \wedge S(\beta) \leq \bar{e}(\alpha \circ \beta, \beta)$. On the other hand, $\bar{e}(\beta, \alpha \circ \beta) \geq \bar{e}(Id_P, \alpha) = S(\alpha)$ (since $Im(\beta) \subseteq P$). By summing up the above results, we get that $S(\alpha) \wedge S(\beta) \leq \bar{e}(\beta, \alpha \circ \beta) \wedge \bar{e}(\alpha \circ \beta, \beta)$. \square

Definition 4.3. Let $(P; e)$ be an L -ordered set, $t \in L$ and $f : (P; e) \rightarrow (P; e)$ be monotone. An element $x \in P$ is called a t -fixpoint of f if $t \leq e(x, f(x)) \wedge e(f(x), x)$. Obviously, the concepts of a 1-fixpoint and a fixpoint are the same.

Corollary 4.4. Consider the assumptions of Theorem 4.2 and let $\beta = \sqcup S$. Then

- (i) For each $x \in P$ and each $f \in H_P$, $\beta(x)$ is a t -fixpoint of f , where $t = S(f) \wedge S(\beta)$.
- (ii) For each $f \in H_P$, $S(f) \leq \bar{e}(f, \beta \circ f)$.
- (iii) For each $f \in H_P$, there exists $u \in P$ such that $S(f) \wedge S(\beta) \leq e(f(u), u) \wedge e(u, f(u))$.
- (iv) If $f \in H_P$ such that $S(f) = 1$, then $f \circ \beta = f$. That is, $\beta(x)$ is a fixpoint for f for all $x \in P$.

Proof. (i) The proof is a straightforward consequence of Theorem 4.2(i).

(ii) Let $f \in H_P$. Since $\beta = \sqcup S$, then

$$\begin{aligned}
S(f) &= \bar{e}(Id_P, f) \leq \bar{e}(\beta, \beta \circ f), \text{ since } \beta \text{ is monotone} \\
&= \bigwedge_{\alpha \in P} (S(\alpha) \rightarrow \bar{e}(\alpha, \beta \circ f)), \text{ by Theorem 2.7(i).}
\end{aligned}$$

So, $S(f) \leq S(f) \rightarrow \bar{e}(f, \beta \circ f)$, which implies that $S(f) \leq \bar{e}(f, \beta \circ f)$ (since L is a frame, $a \leq a \rightarrow b$ implies that $a = a \wedge (a \rightarrow b) = a \wedge b$, so $a \leq b$).

(iii) By (i), for each $x \in P$, the element $u = \beta(x)$ satisfies the condition $S(f) \wedge S(\beta) \leq e(f(u), u) \wedge e(u, f(u))$.

(iv) Let $f \in H_P$ such that $S(f) = 1$. Then by Theorem 2.7(i), $1 = S(f) \leq \bar{e}(f, \beta)$, hence $f(x) \leq \beta(x)$ for all $x \in P$, which implies that $S(\beta) = \bigwedge_{x \in P} e(x, \beta(x)) \geq \bigwedge_{x \in P} e(x, f(x)) = S(f) = 1$. So, by (i), $1 = S(f) \wedge S(\beta) \leq \bar{e}(f \circ \beta, \beta) \wedge \bar{e}(\beta, f \circ \beta)$. That is, $f \circ \beta = \beta$. \square

We know that if $(P; \leq)$ is a CPO and $f : P \rightarrow P$ is an ordered preserving map, then $Fix(f)$ has a least element. Moreover, by Theorem 3.5, if $(P; e)$ is an L -complete lattice, then $\sqcap S_f$ exists and is the least fixpoint of f . In the sequel, we attempt to find conditions for a monotone map on a fuzzy $dcpo$ $(P; e)$ under which $\sqcap S_f$ exists.

Definition 4.5. Let $(P; e)$ be an L -ordered set, $S \in L^P$ and $X \subseteq P$. An element $b \in X$ is called a *join of S in X* and is denoted by $\sqcup_X S = b$ if, for each $x \in X$, $e(b, x) = \bigwedge_{y \in X} (S(y) \rightarrow e(y, x))$. In a similar way, we can define the notion of a *meet of S in X* , $\sqcap_X S$.

Remark 4.6. Let $(P; e)$ be an L -ordered set, $S \in L^P$ such that $a = \sqcup S$ and $X \subseteq P$. Suppose that b is a join of S in X . Then $b \in X$ and for each $x \in X$, $e(b, x) = \bigwedge_{y \in X} (S(y) \rightarrow e(y, x)) \geq \bigwedge_{y \in P} (S(y) \rightarrow e(y, x)) = e(a, x)$ and so $1 = e(b, b) = e(a, b)$. Thus, $a \leq_e b$. By a similar way, we can show that, if $a' = \sqcap S$ and b' is a meet of S in X , then $b' \leq_e a'$.

In a special case, if X is a subset of an L -ordered set $(P; e)$, $S \in L^P$, $b = \bigsqcup_X S$ ($b' = \bigsqcap_X S$) and $a = \sqcup S$ ($a' = \sqcap S$) such that $\text{Supp}(S) := \{x \in P \mid S(x) \neq 0\}$ is a subset of X , then for all $x \in X$, we have

$$e(a, x) = \bigwedge_{y \in P} (S(y) \rightarrow e(y, x)) = \bigwedge_{y \in X} (S(y) \rightarrow e(y, x)) = e(b, x).$$

So, $e(a, b) = e(b, b) = 1$. That is, $a \leq_e b$. A similar proof shows that $b' \leq_e a'$.

Definition 4.7. Let X be a non-empty subset of an L -ordered set $(P; e)$ and $a \in P$. An element $b \in X$ is called a *strong L -cover* for a in X if $e(a, x) = e(b, x)$ for all $x \in X$. We must note that, from $b \in X$ it follows that $e(a, b) = e(b, b) = 1$. Also, if a has a strong L -cover in X , then it is unique. Indeed, if $b, b' \in X$ are strong L -covers for a in X , then by definition, $1 = e(b', b') = e(a, b') = e(b, b')$. Similarly, $e(b', b) = 1$, so $b = b'$.

Proposition 4.8. Let X be a non-empty subset of an L -ordered set $(P; e)$, $S \in L^P$ $a = \sqcup S$ such that $\text{Supp}(S) \subseteq X$. Then b is a strong L -cover for a in X if and only if $b = \bigsqcup_X S$.

Proof. Let $b = \bigsqcup_X S$. Then for all $x \in P$,

$$\begin{aligned} e(b, x) &= \bigwedge_{y \in X} (S(y) \rightarrow e(y, x)) = \bigwedge_{y \in P} (S(y) \rightarrow e(y, x)), \text{ since } \text{Supp}(S) \subseteq X \\ &= e(a, x), \text{ since } \sqcup S = a. \end{aligned}$$

It follows that b is a strong L -cover for a in X . The proof of the converse is similar. \square

Definition 4.9. Let $(P; e)$ be an L -ordered set and $f : P \rightarrow P$ be monotone. An element $S \in L^P$ is called *f -invariant* if, for all $x \in P$, $S(x) \leq S(f(x))$.

Example 4.10. Let $(P; e)$ be an L -ordered set, $f : P \rightarrow P$ be a monotone map and a be a fixpoint of f .

(i) Then the maps $\downarrow a : P \rightarrow L$ and $\uparrow a : P \rightarrow L$ defined by $\downarrow a(x) = e(x, a)$ and $\uparrow a(x) = e(a, x)$ (see [16, Def. 3.15]), respectively are f -invariant. In fact, for each $x \in P$, $\downarrow a(x) = e(x, a) \leq e(f(x), f(a)) = e(f(x), a) = \downarrow a(f(x))$. So $\downarrow a$ is f -invariant. By a similar way, $\uparrow a$ is f -invariant.

(ii) The maps S_f and T_f are f -invariant. Let $x \in P$. Since f is monotone, then

$$S_f(x) = e(f(x), x) \leq e(f(f(x)), f(x)) = S_f(f(x)), \quad T_f(x) = e(x, f(x)) \leq e(f(x), f(f(x))) = T_f(f(x)).$$

Therefore, S_f and T_f are f -invariant.

Theorem 4.11. Let $(P; e)$ be a fuzzy dcpo with zero (that is, there is $0 \in P$ such that $e(0, x) = 1$ for all $x \in P$), $f \in H_P$, $Y = \{a \in P \mid e(a, f(a)) = 1\}$ and $M = \text{Fix}(f)$. If each element of Y has a strong L -cover in M , then $(M; e)$ is a fuzzy sub dcpo of $(P; e)$.

Proof. Clearly, $(P; \leq_e)$ is a CPO, so by [5, Thm. 8.22], f has a fixpoint (or M has a least element). Assume that each element of Y has a strong L -cover. Put a fuzzy directed subset S of $(M; e)$. Define $\overline{S} : P \rightarrow L$, by $\overline{S}(x) = S(x)$ for all $x \in M$ and $\overline{S}(x) = 0$ for all $x \in P - M$. It can be easily shown that \overline{S} is a fuzzy directed subset of $(P; e)$. It follow that $\sqcup \overline{S}$ exists. Let $a = \sqcup \overline{S}$. Then for each $x \in P$,

$$e(a, x) = \bigwedge_{y \in P} (\overline{S}(y) \rightarrow e(y, x)) = \bigwedge_{y \in M} (S(y) \rightarrow e(y, x)) \quad (4.16)$$

and so $1 = e(a, a) = \bigwedge_{y \in M} (S(y) \rightarrow e(y, a)) \leq \bigwedge_{y \in M} (S(y) \rightarrow e(f(y), f(a))) = \bigwedge_{y \in M} (S(y) \rightarrow e(y, f(a))) = e(a, f(a))$, which implies that $a \in Y$. Hence by the assumption, a has a strong L -cover in M , b say, whence $e(a, x) = e(b, x)$ for all $x \in M$. From (4.16) it follows that $e(b, x) = e(a, x) = \bigwedge_{y \in M} (S(y) \rightarrow e(y, x))$. Therefore, $b = \bigcap_X S$ and so $(M; e)$ is a fuzzy *dcpo* with zero. Conversely, let $(M; e)$ be a fuzzy *dcpo*. Put $a \in Y$. Define $S : P \rightarrow L$ by $S(a) = 1$ and $S(x) = 0$ for all $x \in P - \{a\}$. \square

Example 4.12. Let $(P; \leq)$ be a CPO. It can be easily shown that the $\{0, 1\}$ -ordered set $(P; e_{\leq})$ is a fuzzy *dcpo* with zero. Put a monotone map $f : (P; e_{\leq}) \rightarrow (P; e_{\leq})$. Since f is monotone, then for each $a \in P$ satisfying the condition $a \leq f(a)$, we have $A = \{x \in P \mid a \leq x\}$ is a CPO and $f : A \rightarrow A$ is a monotone map, so by [5, Thm. 8.22] f has a least fixpoint on A , b say. Let x be another fixpoint of f .

- (1) If $a \leq x$, then $b \leq x$, so $e_{\leq}(a, x) = e_{\leq}(b, x) = 1$.
- (2) If $a \not\leq x$, then $b \not\leq x$, so $e_{\leq}(a, x) = e_{\leq}(b, x) = 0$.

Hence, b is a strong L -cover for a in $\text{Fix}(f)$, whence $(P; e_{\leq})$ satisfies the conditions of Theorem 4.11.

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